$3D$ Ising and other models from symplectic fermions

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Abstract

We study a model of N complex component symplectic fermions in D spacetime dimensions. It has an infra-red stable fixed point in $2 < D < 4$ dimensions we refer to as $Sp^{(D)}_{2N}$. Based primarily on the comparison of exponents, we conjecture that the critical exponents for the $3D$ Wilson-Fisher fixed point for an $O(N)$ invariant N-component bosonic field can be computed in the $Sp^{(3)}_{-2N}$ theory. The $3D$ Ising model then corresponds to $Sp^{(3)}_{-2}$. To lowest order, the exponent β agrees with known results to 1 part in 1000 and is within current error bars. The ν exponent agrees to 1% and we suggest this is because we only went to 1-loop for this exponent.

I. INTRODUCTION

In $2D$ there exists a vast variety of well-understood critical points, i.e. fixed points of the renormalization group (RG), and for many practical purposes there is a complete classification[1]. In $3D$ the known critical points are comparatively rare. The best known is the Wilson-Fisher[2] fixed point which can be understood as a linear sigma model for an M-component vector field $\vec{n}$ with the euclidean space action:

$$S_{WF} = \int d^D x \left(\frac{1}{2} \partial_\mu \vec{n} \cdot \partial_\mu \vec{n} + \widetilde{\lambda} \, (\vec{n} \cdot \vec{n})^2 \right) \tag{1}$$

where $\partial_\mu \partial_\mu = \sum_{\mu=1}^{D} \partial_{x_\mu}^2$. We will refer to the fixed point theory as $O_M^{(D)}$. It describes classical temperature phase transitions in a variety of magnetic systems.

In our recent work[3], motivated primarily by the search for a deconfined quantum critical point in $2d$ anti-ferromagnetism[4, 5] , we considered a symplectic fermion theory with action:

$$S_\chi = \int d^D x \left(\partial_\mu \chi^\dagger \partial_\mu \chi + 16\pi^2 \lambda \, |\chi^\dagger \chi|^2 \right) \tag{2}$$

where χ is an N-component complex fermionic field, $\chi^\dagger \chi = \sum_{i=1}^{N} \chi_i^\dagger \chi_i$. We refer to this model as a symplectic fermion because in terms of real fields, each component can be written as $\chi = \eta_1 + i\eta_2$, $\chi^\dagger = \eta_1 - i\eta_2$ and the free action is

$$S = i \int d^D x \; \epsilon_{ij} \; \partial_\mu \eta_i \partial_\mu \eta_j \tag{3}$$

where $\epsilon_{12} = -\epsilon_{21}$ and thus has $Sp(2)$ symmetry. The $2N$ real component theory is invariant under $\eta \to U\eta$ where U is a $2N \times 2N$ matrix satisfying $U^T \epsilon_N U = \epsilon_N$ where $\epsilon_N = \epsilon \otimes 1_N$. Thus the complex N-component theory has $Sp(2N)$ symmetry. The symplectic fermion is known to have important applications in physics, for instance to dense polymers[7] in $2D$, and to disordered Dirac fermions in $2+1$ dimensions[8].

It is well-known that the symplectic fermion is non-unitary. In possible applications to $2d$ anti-ferromagnetism, this issue was addressed in [3], so we do not repeat this here since the present context is very different. For statistical mechanics, as we will explain below, the non-unitarity is of a very different nature.

As shown in [3], the $(\chi^\dagger \chi)^2$ interactions drive the theory to a new infrared stable fixed point, which we will refer to as $Sp_{2N}^{(D)}$[23]. In applications to quantum critical spin liquids[3, 6] in $2d$, the χ fields, for $N = 2$, were proposed to describe a deconfined quantum critical point

of an anti-ferromagnetic spin system where the spin field $\vec{n}$ is composite in terms of χ: $\vec{n} = \chi^\dagger \vec{\sigma} \chi$. In the N=2 component case, $\vec{\sigma}$ are just the Pauli matrices. As will become clear, in the present context there will be no need to generalize the above expression for arbitrary N since $\vec{n}$ will not be a composite field.

In this article we focus on applications of the critical theories $Sp_{2N}^{(D)}$ to statistical mechanics in $3D$. The main conjecture is that, for the purpose of calculating the critical exponents, the following two models are equivalent

$$O_M^{(3)} = Sp_{-2M}^{(3)}, \qquad \text{for } M \geq 0 \tag{4}$$

We emphasize that we are not claiming a complete equivalence of the above models, but rather, that some of the $O_M^{(3)}$ model exponents can be calculated in the symplectic fermion theory. This conjecture is based on an argument that $Sp_{2N}^{(3)}$ for negative N should correspond to $O_M^{(3)}$ models, however not according to the naive equivalence $N = -M/2$. (See below for explanations.) We then conjecture $N = -M$ based mainly on the agreement of the exponents and an additional symmetry argument. Since the Ising model is known to be in the universality class of $O_1^{(3)}$, we compute $3D$ Ising model exponents, and exponents for other models as well, and show that to lowest order they agree exceedingly well with known results.

II. SCALING THEORY AND RG ANALYSIS

In this section we summarize the results of the RG analysis in [3]. The beta function, computed to 1-loop only, for the N component model in D space-time dimensions is

$$\frac{d\lambda}{d\ell} = (4 - D)\lambda + (N - 4)\lambda^2 \tag{5}$$

where increasing the length e^ℓ corresponds to the flow toward low energies. The above beta function has a zero at

$$\lambda_* = \frac{4 - D}{4 - N} \tag{6}$$

Note that λ_* changes sign at $N = 4$. It is not necessarily a problem to have a fixed point at negative λ since the particles are fermionic: the energy is not unbounded from below because of the Fermi sea. Near λ_* one has that $d\lambda/d\ell \sim (D - 4)(\lambda - \lambda_*)$ which implies the fixed point is IR stable regardless of the sign of λ_*, so long as $D < 4$.

For some exponents we need a measure of the departure from the critical point; these are the parameters that are tuned to the critical point in simulations and experiments:

$$S_\chi \to S_\chi + \int d^D x \; (m^2 \chi^\dagger \chi) \tag{7}$$

Above, m is a mass. The correlation length exponent ν is then defined as

$$\xi \sim m^{-\nu} \tag{8}$$

To streamline the discussion, let $[\![X]\!]$ denote the scaling dimension of X in energy units, including the non-anomalous classical contribution which depends on D. An action S necessarily has $[\![S]\!]$ =0. Using $[\![d^D\mathbf{x}]\!] = -D$, the classical dimensions of couplings and fields are determined from $[\![S]\!] = 0$. The exponents are functions of the anomalous dimensions γ_χ, γ_m of χ and m:

$$[\![\chi]\!] \equiv (D-2)/2 + \gamma_\chi, \qquad [\![m]\!] \equiv 1 - \gamma_m \tag{9}$$

The γ_χ exponent determines the two point function of the χ fields:

$$\langle \chi^\dagger(\mathbf{x})\chi(0)\rangle \sim \frac{1}{|\mathbf{x}|^{D-2+2\gamma_\chi}} \tag{10}$$

Since $[\![\xi]\!] = -1$, this implies $\nu = -[\![\xi]\!]/[\![m]\!] = 1/[\![m]\!]$. Let us also define the exponent $\beta = 2[\![\chi]\!]/[\![m]\!]$:

$$\frac{1}{\nu} = 1 - \gamma_m, \qquad \beta = \frac{D-2+2\gamma_\chi}{1-\gamma_m} \tag{11}$$

The normalization of the above exponents is appropriate for the context of deconfined quantum criticality, and will be related to conventional normalizations of the exponents of O_M models in the next section.

In [3], γ_m was computed to 1-loop and γ_χ to two-loops. The computation can be understood as essentially the lowest order epsilon expansion around $D = 4$. We found the following χ field exponents for any N, D:

$$\gamma_m = \frac{(4-D)(1-N)}{2(4-N)}, \qquad \gamma_\chi = \frac{(4-D)^2(1-N)}{4(4-N)^2} \tag{12}$$

These in turn imply the following:

$$\begin{aligned} \nu &= \frac{2(4-N)}{(2-D)N + D + 4} \\ \beta &= \frac{2(D-2)(N^2-4N+12) + D^2(1-N)}{(4-N)(D(1-N)+2(N+2))} \end{aligned} \tag{13}$$

III. $3D$ STATISTICAL MECHANICS

Since we are now mainly interested in the context of statistical mechanics in D spacial dimensions, we need to reconsider the definitions of the exponents in order to compare with what is known about the $O_M^{(3)}$ models. In analogy with Ising models, consider an action

$$S = S_{\text{critical}} + \int d^D x \; t \, \Phi_\epsilon(x) \tag{14}$$

where $t = (T - T_c)/T_c$, and Φ_ϵ is an 'energy operator'. The exponent η_{stat} is defined by the two point function of the spin field $\sigma(x)$:

$$\langle \sigma(\mathbf{x}) \sigma(0) \rangle = \frac{1}{|\mathbf{x}|^{D-2+\eta_{\text{stat}}}} \tag{15}$$

and the exponents $\nu_{\text{stat}}, \beta_{\text{stat}}$ as follows:

$$\xi \sim t^{-\nu_{\text{stat}}}, \qquad \langle \sigma \rangle \sim t^{\beta_{\text{stat}}} \tag{16}$$

The spin field is identified as the M-vector $\vec{n}$. It is well-known that the above exponents satisfy the scaling relation:

$$\beta_{\text{stat}} = \nu_{\text{stat}} (D - 2 + \eta_{\text{stat}})/2 \tag{17}$$

As explained below, the above exponents are related to ν, β of the previous section by $\nu_{\text{stat}} = \nu/2$ and $\beta_{\text{stat}} = \beta/4$.

Let $O_M^{(D)}$ denote the Wilson-Fisher fixed point. First let us point out that there is a simple argument that $O_M^{(D)}$ should correspond to $Sp_{-M}^{(D)}$. We emphasize that our conjecture eq. (4) differs by a 2. The naive argument goes as follows. Suppose we try to make the functional integral over χ gaussian by introducing an auxiliary field $u(x)$ and the action $S = S_{\text{free}} + \int d^D x \, (u \, \chi^\dagger \chi - \lambda^{-1} u^2)$. Then the functional integral over $\chi^\dagger, \chi$ gives a u-dependent functional determinant to the N-th power. On the other hand, if χ were a complex boson, one would obtain the same determinant to the $(-N)$-th power. Since the O_M model has M real bosonic components, then one naively expects the previously mentioned equivalence. One reason we have to doubt this naive equivalence is that it is not obvious that one can compute correlation functions of the χ's from the effective action for u since u is essentially the composite field $\chi^\dagger \chi$. Below, we will compare the β exponent for the $O_M^{(D)}$ and $Sp_{2N}^{(D)}$ models and show that they do not coincide under the equivalence $N = -M/2$. It also seems likely that at finite temperature the naive equivalence $N = -M/2$ will be spoiled by the

different boundary conditions (periodic verses anti-periodic) for the χ verses $\vec{n}$ fields which are needed to compute the determinant.

Another, though related, approach to identifying our models at negative N is based on Parisi-Sourlas supersymmetry[9]. However as we now explain this would give the same identification as the considerations of the last paragraph. For a recent discussion of this in the context of $2D$ non-linear sigma models see[10]. In this construction, one considers both bosonic and fermionic fields:

$$\vec{\phi} \equiv (\phi_1, \cdots, \phi_{M+2N}, \eta_1, \cdots \eta_{2N}) \tag{18}$$

where ϕ_a are bosonic, η_i are fermionic, and both are real. The bilinear $\vec{\phi} \cdot \vec{\phi} = \sum_a \phi_a \phi_a + \sum_{i,j} (\epsilon_N)_{ij}\ \eta_i \eta_j$ has $OSp(M+2N|2N)$ symmetry and one can consider a linear sigma model with $(\vec{\phi} \cdot \vec{\phi})^2$ interactions. The Parisi-Sourlas arguments suggest that exponents should only depend on M, i.e. are independent of N, at least if this symmetry is preserved. The purely bosonic $N = 0$ model has O_M symmetry where $\vec{\phi} \sim \vec{n}$, whereas the purely fermionic model $M + 2N = 0$ has Sp_{2N} symmetry. Chosing $M + 2N = 0$, one naively has the equivalence $O_M \sim Sp_{-M}$, which again differs by a factor of two from our conjectured equivalence. The observations of the last paragraph still apply, and we again point out that our two-loop calculations for β appear to contradict the supersymmetry argument.

Since our conjecture $O_M \sim Sp_{-2M}$ differs from the above $O_M \sim Sp_{-M}$, clearly some new and non-obvious arguments are needed in support of it, which we now attempt to provide. The main evidence comes from comparison with exponents. As we show below, the lowest order exponents agree for Sp_{-2M} agree exceedingly well with the very high order epsilon expansion of the O_M model. (See the table below.) For instance, for the Ising model, β is known to be approximately $1.303 \pm .006$ whereas the Sp_{-2} model gives $\beta = 13/10 = 1.300$. The Sp_{-1} model on the other hand ($N = -1/2$) gives $\beta = 56/45 = 1.244$.

A symmetry argument supporting our conjecture can be given as follows. First observe that the $Sp(2N)$ symmetry of the symplectic fermions has an $O(N)$ subgroup, generated by $1 \otimes A$ where A is an $N \times N$ antisymmetric matrix. Near $D = 2$, the fixed points are expected to be described by a non-linear sigma model. In terms of the χ-fields, the non-linear constraint is $\chi^\dagger \chi = 1$. As before, expressing each component of χ in terms of real components η, $\chi_1 = \eta_1 + i\eta_2$, $\chi^\dagger = \eta_1 - i\eta_2$, etc, then if η is fermionic satisfying $\eta_i^2 = 0$, $\forall i$,

the constraint reads

$$\eta_1\eta_2 + \eta_2\eta_3 + \cdots + \eta_{2N-1}\eta_{2N} = 1 \tag{19}$$

It is evident that this constraint has the above-mentioned $O(N)$ symmetry. On the other hand, if the η were bosonic, the constraint reads $\sum_{i=1}^{2N} \eta_i^2 = 1$, which has an $O(2N)$ symmetry. Thus, based only these symmetry arguments, the $O(N)$ models could have realizations in the symplectic fermion theory Sp_{-2N}. The flip in sign in Sp_{-2N} is then understood as coming from the need to flip the statistics of χ.

Another, somewhat different, argument comes from $2D$. What we can learn from the well-understood $2D$ case is somewhat limited because the phase structure is quite different than $D > 2$[11]. Nevertheless, we can give some additional support for our conjecture as follows. The $O(M)$ models are only critical for $-2 < M < 2$ in $2D$ where they are equivalent to loop models. Parameterizing M as $M = -2\cos\pi g$, the Virasoro central charge is known to be $c = 1 - 6(g-1)^2/g$. The region $0 < g < 1$ is referred to a dense loops, and $g > 1$ as dilute. First note that $g = 2$ in the dilute phase corresponds to $M = -2$ and has $c = -2$. This value of c is known to correspond to an $N = 1$ component symplectic fermion, thus here one observes $O_{-2} \sim Sp_2$, in accordance with standard Parisi-Sourlas symmetry. However there exists another interpretation in the dense phase that coincides with our conjecture. The O_M models in $2D$ were studied using supersymmetry in [12, 13]. It was argued in [13] that the dense loop phase is not generic, i.e. allowing crossings of the loops leads to a dangerously irrelevant perturbation which can drive it to a distinct massless phase, referred to as a Goldstone phase. Arguments were also given that this Goldstone phase is more generic and what one expects in $D > 2$. The case $OSp(1|2)$, which corresponds to O_{-1}, is a good example since both $M, N = 1$ are positive, and was studied in detail. $M = -1$ is realized in the dense loop phase at $g = 1/3$. The Goldstone phase is expected to have massless degrees of freedom described by $OSp(1|2)/OSp(0|2) \sim \mathbf{S}^{0|2}$ where $\mathbf{S}^{0|2}$ is the supersphere. The non-linear sigma model has the action

$$S = \frac{1}{\lambda}\int d^2x \left((\partial_\mu\phi)^2 + i\partial_\mu\eta_1\partial_\mu\eta_2\right) \tag{20}$$

where ϕ is a single component real field and the sigma model constraint is $\phi^2 + \eta_1\eta_2 = 1$. Using the Grassman nature of the η's, the constraint can easily be solved as $\phi = 1 - \frac{1}{2}\eta_1\eta_2$.

This leads to the action

$$S = \frac{1}{\lambda} \int d^2x \left(i\partial\eta_1\partial\eta_2 - \frac{1}{2}\eta_1\eta_2\partial\eta_1\partial\eta_2 \right) \tag{21}$$

In $D > 2$ the interaction is irrelevent and one ends up with a single $N = 1$ complex component symplectic fermion with $c = -2$. This value of c was verified with the Bethe ansatz in [12]. So in this case, $O_{-1} \sim Sp_2$, which coincides with our conjecture eq. (4) extended to negative M.

It should be clear by now that if the explanation of our conjecture is based on the above kinds of arguments, then computing exponents for O_M models in an Sp_{-2M} model is simply a trick and one need not be concerned about the non-unitarity of the symplectic fermion.

The above arguments are not a complete explanation and the equivalence conjectured in the introduction was originally based on the comparison of exponents. Let us identify Φ_ϵ as $\chi^\dagger\chi$. Because of the m^2 in eq. (7) one finds

$$\nu_{\rm stat} = \nu/2 \tag{22}$$

Supersymmetry implies that $[\![\vec{n}]\!] = [\![\chi]\!]$ since they are both in the same multiplet $\vec{\phi}$, thus

$$\beta_{\rm stat} = \frac{[\![\chi]\!]}{2[\![m]\!]} = \beta/4 \tag{23}$$

The exponent $\eta_{\rm stat}$ then follows from the scaling equation (17), which imples $\eta_{\rm stat} = 2\gamma_\chi$.

In $3D$ the formulas of section II give

$$\nu = \frac{2(4-N)}{7-N}, \qquad \beta = \frac{2N^2 - 17N + 33}{N^2 - 11N + 28} \tag{24}$$

Note that eqns. (22,23) in the $N \to -\infty$ limit give $\nu_{\rm stat} \to 1$ and $\beta_{\rm stat} \to 1/2$ which is consistent with the $M \to \infty$ limit of the O_M model and also explains the factor of 4 in eq. (23).

The equivalent lowest order epsilon-expansion result in the bosonic $O_M^{(3)}$ model is[14, 15, 16]

$$\nu = \frac{2(M+8)}{M+14}, \qquad \beta = \frac{2M^2 + 33M + 130}{M^2 + 22M + 112} \qquad (O_M) \tag{25}$$

The two sets of expressions eqs. (24,25) both come from identical Feynman diagrams. Note that the ν exponents of $O_M^{(3)}$ and $Sp_{2N}^{(3)}$ agree with the identification $N = -M/2$. This is to be expected since to this lowest order this exponent only involves the 1-point function of

$\chi^\dagger\chi$ and doesn't involve a 2-point function of χ-fields. The exponents β on the other hand do not agree under this identification.

In the table below we summarize some of the above formulas and compare them with the highest order epsilon expansions available to date. The column "$O_M^{(3)}$ high − eps." is the Borel-summed 7-loop epsilon expansion for the $O_M^{(3)}$ model, taken from [17][24]. The column "$O_M^{(3)}$ low − eps." is the lowest order epsilon expansion eq. (25).

$O_M^{(3)}$		$Sp_{-2M}^{(3)}$	$O_M^{(3)}$ high-eps.	$O_M^{(3)}$ low-eps.
$M=0$	ν	$8/7 = 1.143$	$1.176 \pm .002$	$8/7 = 1.143$
	β	$33/28 = 1.178$	$1.210 \pm .003$	$65/56 = 1.161$
3D Ising	ν	$5/4 = 1.250$	$1.261 \pm .002$	$6/5 = 1.200$
$M=1$	β	$13/10 = 1.300$	$1.303 \pm .006$	$11/9 = 1.222$
$M=2$	ν	$4/3 = 1.333$	$1.341 \pm .003$	$5/4 = 1.250$
	β	$25/18 = 1.388$	$1.388 \pm .006$	$51/40 = 1.275$
$M=3$	ν	$7/5 = 1.400$	$1.415 \pm .007$	$22/7 = 1.294$
	β	$51/35 = 1.457$	$1.465 \pm .01$	$247/187 = 1.321$
$M=4$	ν	$16/11 = 1.454$	$1.482 \pm .012$	$4/3 = 1.333$
	β	$133/88 = 1.511$	$1.532 \pm .018$	$49/36 = 1.361$

The agreement of the first two columns is rather striking, but still, a few comments are in order. Most of our β exponents are within error bars. For the Ising case, β is within error bars and accurate to 1 part in 1000. The ν exponents, though very reasonable, are not as accurate, and this may be due to the fact that we only calculated it to 1-loop, whereas β required a 2-loop calculation. (The RG beta-function is still only at 1-loop, however the β exponent depends on γ_χ and the lowest contribution to the latter arises at 2 loops[3].) For the Ising model, much more is known, and with smaller errors, since it has been studied in a variety of Monte-Carlo simulations[18, 19, 20, 21]. These works obtained $\nu = 1.2596, 1.2612, 1.2602, 1.2588$ respectively, all with errors of about $\pm(.001 - .002)$. A non-perturbative RG approach gives $\nu = 1.264$[22]. So our 1-loop calculation of ν is off by about 1% in the Ising case. We emphasize that this does not yet spoil our conjecture on the equivalence $O_M^{(3)} = Sp_{-2M}^{(3)}$, rather it suggests that one must go to higher loop order to be within the error bars of what is already known.

IV. CONCLUSIONS

The excellent agreement of the exponents of the $O_N^{(3)}$ and $Sp_{-2N}^{(3)}$ models certainly strongly supports our conjecture. We also gave some arguments, though incomplete, towards an explanation based on symmetries, in particular the fact that $O(N)$ is a subgroup of $Sp(2N)$. Because the ν exponent is so well measured for the Ising model, it appears our exponent is off by 1% in that case, but we suggested that this could be due to the fact that we only went to 1-loop for ν. The β exponent on the other hand, which involves the 2-loop calculation of γ_χ, is within the small error bars. Our calculation was a quite simple 1- and 2-loop calculation, so, if our conjecture is correct, the symplectic fermion is at least a very efficient way to compute exponents compared to the delicate Borel summations one has to do in the very high order epsilon expansions in the bosonic description.

Higher order calculations will certainly change the exponents, so one may worry that the agreement so far is too good. However we wish to point out that for the O_M models, it is known that when one goes to higher orders, the agreement becomes worse, which is why one needs to Borel sum. Higher order calculations are currently in progress and will be reported in a future publication.

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[23] In an earlier version of this article we referred to the fixed point as $Sp_N^{(D)}$. Here, 'N' has the same meaning as before, however we follow the conventional notation for the $Sp(2N)$ Lie group.

[24] We present our exponents in the conventions appropriate for quantum spin liquids, which, as explained, differ by factors of two and four from the definitions in [17]. See eqs. (22,23).